

DISCRETE MATH BACKGROUND

TOPICS:

1. Sets.
2. Relations.
3. Functions.

1 Sets

Definition 1 (*Set*)

A set is a collection of distinct objects, i.e. no duplicates, e.g. $\{1, 2, 3, a, b\}$. The objects are called elements, or members (same meaning).

Three key concepts for sets are:

1. set membership

$$\begin{aligned}x \in S & : x \text{ is a member of } S \\x \notin S & : x \text{ is not a member of } S\end{aligned}$$

e.g.

$$\begin{aligned}1 & \in \{1, 2, 3, a, b\} \\d & \notin \{1, 2, 3, a, b\}\end{aligned}$$

2. subset

$A \subseteq B$: every member of A is also a member of B , or

$$\text{if } x \in A \text{ then } x \in B$$

e.g.

$$\begin{aligned}\{1, a\} & \subseteq \{1, 2, 3, a, b\} \\ \{1, d\} & \not\subseteq \{1, 2, 3, a, b\}\end{aligned}$$

3. equality

$A = B$: A and B have the same members

$$\text{if } x \in A \text{ then } x \in B$$

and

$$\text{if } x \in B \text{ then } x \in A$$

$=$ and $\subseteq$ are related as follows:

$$A = B \text{ exactly when } A \subseteq B \text{ and } B \subseteq A$$

1.1 Specifying sets

We often specify a set using the following form

$$\{x : p(x)\}$$

where $p(x)$ is some condition on x , e.g.

$$\{x : x = 2y \text{ for some number } y\}$$

is the set of even numbers. Thus $\{x : p(x)\}$ is the set of all x for which $p(x)$ is true.

1.2 Set operations

Let A, B be sets. Then we define: Set intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Set union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Set difference: $A - B = \{x : x \in A \text{ and } x \notin B\}$

1.3 Set laws

Let U be a fixed set which we call a “universe”. The universe contains all the elements that we are concerned with, e.g., all the possible states of a program.

Let $A \subseteq U$. Then define $\overline{A} = U - A$.

Set laws.

- Empty set laws

$$A \cap \emptyset = \emptyset$$

$$A \cup \emptyset = A$$

- Idempotency

$$A \cap A = A$$

$$A \cup A = A$$

- Complement

$$\overline{\overline{A}} = A$$

$$A \cap \overline{A} = \emptyset$$

$$A \cup \overline{A} = U$$

- Commutative

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

- Associative

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

- Distributive

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- Absorption

$$A \cap (A \cup B) = A$$

$$A \cup (A \cap B) = A$$

- DeMorgan

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

1.4 Partition

Let S be any set. Then, the sets $S_1, \dots, S_n$ form a partition of S iff

1. for all i, j such that $1 \leq i, j \leq n$ and $i \neq j$: $S_i \cap S_j = \emptyset$, i.e., $S_1, \dots, S_n$ are pairwise disjoint
2. $\bigcup_{1 \leq i \leq n} S_i = S$, i.e., the union of $S_1, \dots, S_n$ is S .
3. for all i such that $1 \leq i \leq n$: $S_i \neq \emptyset$, i.e., the S_i are all non-empty.

1.5 Ordered pair

The ordered pair (x, y) consists of the elements x and y in the order x first, y second. Thus, $(x, y) \neq (y, x)$ unless $x = y$. We define ordered pairs in terms of sets as follows:

$$(x, y) = \{x, \{x, y\}\}$$

1.6 Cartesian product

Binary Cartesian product.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Examples.

Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$. Then

$$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$$

$$B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$$

In general, $A \times B \neq B \times A$

n -ary Cartesian product.

$$A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) : a_1 \in A_1 \text{ and } \cdots \text{ and } a_n \in A_n\}$$

2 Relations

Definition 2 (*Relation*)

A binary relation R on sets A and B is a subset of $A \times B$. A binary relation R on a set A is a subset of $A \times A$.

We sometimes write aRb for $(a, b) \in R$.

Let $R \subseteq A \times A$. Then:

- R is reflexive iff for all $a \in A : aRa$
- R is symmetric iff for all a, b such that $a \in A$ and $b \in A$: aRb implies bRa
- R is transitive iff for all a, b, c such that $a \in A$ and $b \in A$ and $c \in A$: $(aRb \text{ and } bRc)$ implies aRc
- R is antisymmetric iff for all a, b such that $a \in A$ and $b \in A$: $(aRb \text{ and } bRa)$ implies $a = b$

Examples.

Let $A = \{x, y, z, w\}$. Then

$R = \{(x, x), (y, y)\}$ is not reflexive

$R = \{(x, x), (y, y), (z, z), (w, w)\}$ is reflexive

$R = \{(x, x), (y, y), (z, z), (w, w), (x, y)\}$ is reflexive

$R = \{(x, x), (y, y)\}$ is symmetric

$R = \{(x, x), (y, y), (z, z)\}$ is symmetric

$R = \{(x, x), (y, y), (z, z), (x, y)\}$ is not symmetric

$R = \{(x, y), (y, x)\}$ is symmetric

$R = \{(x, y), (y, z)\}$ is not transitive (needs (x, z))

$R = \{(x, y)\}$ is transitive

$R = \{(x, y), (z, w)\}$ is transitive

$R = \{(x, x), (y, y)\}$ is antisymmetric

$R = \{(x, x), (y, y), (z, z), (w, w)\}$ is antisymmetric $R = \{(x, y), (y, z)\}$ is not antisymmetric

The relationship between symmetry and antisymmetry:

If R contains at least one pair (x, y) with $x \neq y$, and R is symmetric, then R cannot be antisymmetric.

This is because symmetry requires that $(y, x) \in R$. Now antisymmetry requires $x = y$, which contradicts that $x \neq y$. Otherwise, there is no relationship between symmetry and antisymmetry, as the following examples show:

3 Functions

Definition 3 (*Function*)

A function f from set A to set B is a binary relation on $A \times B$ (i.e. a subset of $A \times B$) such that:

1. $\forall(a : a \in A : \exists(b : b \in B : (a, b) \in f))$
i.e., for every $a \in A$ there is at least one $b \in B$ such that $(a, b) \in f$
2. $\forall(a, b, c : a \in A \wedge b \in B \wedge c \in B : (a, b) \in f \wedge (a, c) \in f \Rightarrow b = c)$
i.e., for every $a \in A$ there is at most one $b \in B$ such that $(a, b) \in f$

Thus, for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$. We use $f(a)$ to denote this b . a is the argument of f and b is the value of f .

We write $f : A \mapsto B$ to indicate that f is a function from A to B . A is the domain of f , and B is the codomain of f .

Let $f : A \mapsto B$, and let $A' \subseteq A$. Then define

$$f(A') = \{b : \exists(a : a \in A' : b = f(a))\}$$

$f(A')$ is the image of A' .

Let $f : A \mapsto B$:

f is a surjection, or onto, iff

$$\forall(b : b \in B : \exists(a : a \in A : f(a) = b))$$

f is an injection, or one-to-one, iff

$$\forall(a, a' : a \in A \wedge a' \in A : a \neq a' \Rightarrow f(a) \neq f(a'))$$

f is a bijection iff f is both: a surjection and an injection