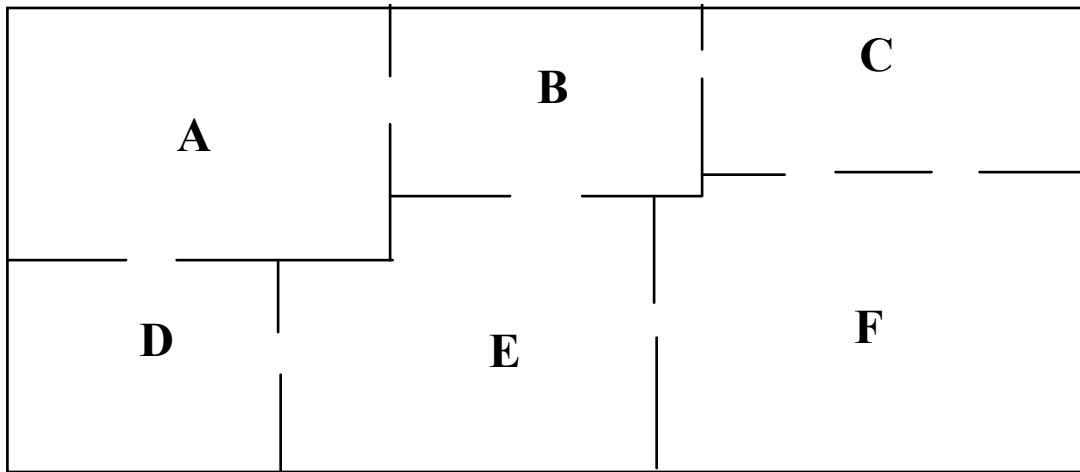


Applications Of Matrices

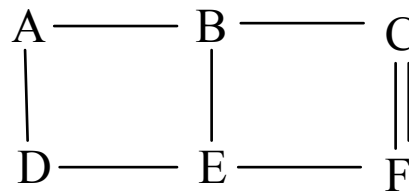
**2004 NCSSM
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The maze below was built as a test container for rats.



a) The figure to the right represents a graph for this maze in which each room represents a vertex and each door represents an edge. If there had been one-way doors, they would be represented with directed edges.



d) Construct an adjacency matrix for this maze and find the cube of this matrix.

adjacency matrix

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

square of the adjacency matrix

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

cube of the adjacency matrix

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

Enter the adjacency matrix in your calculator. Square and cube it and record the results above. Examine the matrices and give a practical application for what the numbers in these matrices might represent.

Markov Chain Application

We are now going to find out what happens when a rat is placed in the original maze. Assume that each minute, the rat always moves to another room through a door (assuming the door is not a one-way door against him). For example, if the rat is in room A, he has a $\frac{1}{2}$ chance of going to B, he has a $\frac{1}{2}$ chance of going to D; if the rat is in room F, he has a $\frac{2}{3}$ chance of going to C, and a $\frac{1}{3}$ chance of going to room E. He would have a 0 chance of going to any other room. On the basis of these probabilities, construct a transition matrix for this maze (where a transition represents the movement of the mouse from one room to another) and enter it in your calculator.

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

b) Suppose the rat started in room C. Where would he be after 1 minute? In other words, calculate the probability of his being in each room at the end of 1 minute.

d) If the rat starts in B, where would he be at the end of 1 hour. (60 minutes) (Show what calculation is needed.)

c) What matrix could be used to represent the rat starting in room B? (Call this matrix B.)

e) If the rat starts in Room E, where would he be at the end of one hour? What generalization do you think you can make.

Find a stable state matrix for this Markov chain. Write a possible interpretation for this stable state matrix.

Adjacency matrices and Airlines

You have been hired by the website travelocity.com to help people plan trips between various cities. Often your customers are business travelers so that they want to travel between cities in the morning to conduct a days business. Large cities often provide flights to many cities, but small cities often are quite limited in the number of cities that they service. Your customers are particularly interested in travel between the following cities. Albany, **Boston**, **New York**, **Philly**, **Wash**, **Richmond**, **Detroit**, and **Las Vegas**. For simplicity, we will only use the first letter to refer to the city. Here is the flight information that you are given.

From Boston there are flights to	N, P, W, D
From Albany there are flights to	N, W
From New York there are flights to	B, P, W, R, D, L
From Philly there are flights to	N, B, W, R
From Wash there are flights to	B, A, N, R, P, L
From Richmond there are flights to	N, P, W
From Detroit there are flights to	B, N
From Las Vegas there are flights to	N, W

- 1) Make a graph of this information where vertices represent cities and every edge represents a flight.
- 2) Examine the information to see if there is a round trip passage between every city that is connected by a flight. How would you indicate a flight that goes from one city to another, but does not return?
- 3) An adjacency matrix for the graph is listed below. Discuss the symmetry in the graph and the main diagonal.

	Bos	Alb	NY	Philly	Wash	Rich	Det	Las Veg
Bos	0	0	1	1	1	0	1	0
Alb	0	0	1	0	1	0	0	0

Square of Matrix							
Bos	Alb	NY	Philly	Was	Rich	Det	Las Veg
				h			

NY	1	0	0	1	1	1	1	1
Philly	0	0	1	0	1	1	0	0
Wash	1	1	1	1	0	1	0	1
Rich	0	0	1	1	1	0	0	0
Det	1	0	1	0	0	0	0	0
Las Veg	0	0	1	0	1	0	0	0

4) Enter the matrix in your calculator and square it. Write the square of the matrix in the space provided to the right of the matrix. What does this matrix tell you? Give an explanation of how this information might be valuable to airlines.

5) What might the cube of the matrix represent?

Introduction to Markov chains

An important application of matrices involves combining matrix multiplication with probability. In these problems, we describe a transition from one state to another in terms of probabilities. The game we played in class of tossing a ball between four people is an example, but we will see other more important examples that are important to certain kinds of businesses. A look at last week's game should allow us to examine certain important characteristics of Markov chains.

The assumptions that we made last week: Each person threw to the others according to the following probabilities.

Thrower	Receiver	Probability	Thrower	Receiver	Probability
Jackie	to Brinkley	.25	Chris	to Jackie	.25
Jackie	to Chris	.50	Chris	to Brinkley	.25
Jackie	to Mark	.25	Chris	to Mark	.50
Brinkley	to Mark	.40	Mark	to Brinkley	.10
Brinkley	to Chris	.20	Mark	to Mark	.50
Brinkley	to Jackie	.40	Mark	to Jackie	.20
			Mark	to Chris	.20

Some important characteristics of Markov chain problems is that since they are based on probabilities, the sum of each person's probabilities must equal 1. A picture of this game would look like the diagram below.

We can create a matrix, called a transition matrix to describe the probabilities expressed in the diagram.

$$T = \begin{matrix} & \begin{matrix} J & B & C & M \end{matrix} \\ \begin{matrix} J \\ B \\ C \\ M \end{matrix} & \begin{bmatrix} 0 & .25 & .50 & .25 \\ .40 & 0 & .20 & .40 \\ .25 & .25 & 0 & .50 \\ .20 & .20 & .10 & .50 \end{bmatrix} \end{matrix}$$

Note that entry T_{31} means the probability that Chris throws to Jackie is 25%, and that entry T_{13} means that the probability that Jackie throws to Chris is 50%. Enter this matrix in matrix B.

We will use this transition matrix to try to predict who will have the ball after a certain number of throws. Suppose that Jackie starts with the ball. We can indicate this with the following matrix.

$$A_0 = \begin{matrix} & \begin{matrix} J & B & C & M \end{matrix} \\ \begin{matrix} J \\ B \\ C \\ M \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad \text{Enter this matrix into matrix A.}$$

1) What would the starting matrices be in each of the following cases.

a) The ball starts with Brinkley.

b) The ball starts with Mark.

c) We toss a coin between Chris and Jackie to see where it starts.

d) We start it randomly at one of the four players.

As we multiply these matrices, be sure the mode is set to 2 decimal places.

2) Multiply A_0 times B and record the result.

Explain why you obtained a 1 by 4 matrix.

Explain why the matrix had the values it does.

3) Change the matrix A_0 to indicate that Brinkley starts with the ball. Predict what you will get when you multiply A_0 times B and confirm that you are correct.

4) The most interesting application of Markov chains is to use them to predict what will happen in the long run, in other words, after many throws of the ball. In other words, can we determine the probability of each person ending up with the ball.

Process: Multiply $[A][B]$. then multiply ANS $[B]$. keep pressing enter to iterate this process. What do you observe about the long term behavior?

Record the stable matrix that results from this process and explain specifically what it means.

More Markov Chain examples

Example 1:

In the 60's the American car market was dominated by US made car. The Japanese auto makers, Toyota, Nissan, and Honda were just beginning to make inroads to the American market while the European makers and other countries had a small steady share. Still, most people who owned American, replaced their car with another US made car, while the majority of Japanese car owners replaced their car with another Japanese make. The matrix to the left below represents a possible survey of 500 car owners and their preferences. For example, of 300 US car owners, 240 or 80% planned to replace their car with another US made car, while 45 owners (or 15%) planned to switch to a Japanese car.

The percent matrix to the right represents a transition matrix for a Markov chain.

	US	Jap	Eur	Other	Total
US	240	45	12	3	300
Jap	20	65	10	5	100
Eur	8	8	60	4	80
Other	4	4	2	10	20

	US	Jap	Eur	Other	Total
US	0.8	0.15	0.04	0.01	1.00
Jap	0.2	0.65	0.1	0.05	1.00
Eur	0.1	0.1	0.75	0.05	1.00
Other	0.2	0.2	0.1	0.5	1.00

Suppose that the proportion of cars sold in the US in the mid 60's followed the proportions in the survey. An initial matrix for these proportions would be

	US	Jap	Eur	Other
Prop [	.60	.20	.16	.04]

Suppose a transition can be said to occur every 5 years. Over the course of 20 years (4 transitions), what will happen to the proportion of cars if we use the transition matrix that we computed above. Before doing this on your calculator, estimate what the proportions will be for each country represented.

Coke/Pepsi example

Coke and Pepsi are continuously involved in a competitive struggle to obtain market share from each other. Each company tries to maintain a loyal customer base and hopes to maintain there own customers while attracting customers from their competitor. As a result, a marketing strategy that is used by both companies is to vary the price charged in supermarkets, often by a considerable amount. For example, the price of a 2-liter coke at Ukrops for four consecutive weeks over the last month was \$.99, \$1.49, \$1.29, and \$.79. The price of a 2-liter Pepsi varied in a similar way, but was usually low when coke was high and high when coke was low. Based on independent research surveys, Coke determined the following tendencies of people each month.

94% of Coke drinkers one week stay with Coke the next week.

6% of Coke drinkers switch to Pepsi the next week.

9% of Pepsi drinkers switch to Coke the next week.

91% of Pepsi drinkers stay with Pepsi.

a) Complete the transition matrix for these facts.

b) Suppose that statistics show that last month 72% of the Coke or Pepsi drinkers drink Coke. Write a starting matrix to represent this fact.

$$T = \begin{matrix} & \begin{matrix} \text{Coke} \\ \text{Pepsi} \end{matrix} \\ \begin{matrix} \text{Coke} \\ \text{Pepsi} \end{matrix} & \end{matrix}$$

c) What is the stable state matrix for this problem, and what does it tell you?

d) Pepsi believes that by advertising in the Super Bowl, they can change the Coke drinkers habits so that 12% of the Coke will switch to Pepsi, up from 6%. What effect will this have on the market share for Pepsi?

e) Explain what this tells you about why a company might be willing to spend up to \$2 million for a 30 second ad at the Super Bowl.

Football rankings

The matrix below represents the results of the competition between the ACC schools in football in the 2002 season. How would you rank these teams from 1 to 9 based on these results.

	ACC Football 2002										
	UNC	UVA	NC St.	FI St.	Ga Tc	Wake	Mary	Clem	Duke		Wins
UNC	0	0	0	0	0	0	0	0	1		1
UVA	1	0	1	0	0	1	1	1	1		6
NC St.	1	0	0	1	0	1	0	1	1		5
Florida St.	1	1	0	0	1	1	1	1	1		7
Georgia Tech	1	1	1	0	0	0	0	0	1		4
Wake Forest	1	0	0	0	1	0	0	0	1		3

Maryland	1	0	1	0	1	1	0	1	1	6
Clemson	1	0	0	0	1	1	0	0	1	4
Duke	0	0	0	0	0	0	0	0	0	0
Losses	7	2	3	1	4	5	2	4	8	

Matrix Multiplication on Excel.

To multiply two matrices using excel, begin by entering the matrices. To multiply matrices, the columns of the first matrix must match the rows of the second matrix. The product will have the same number of rows as the first matrix and the same number of columns as the second matrix. When I use a spreadsheet to iterate Markov Chains, I have the first matrix be the starting row matrix, and the second matrix be the transition matrix. You must be careful to enter the transition matrix as a constant (absolute cell references).

To do the matrix multiplication, first highlight the destination for the product, being sure to highlight the appropriate number of rows and columns.

Enter =mmult(

Select the first matrix, type a comma, and select the second matrix (making it constant if necessary). Close the parentheses and while holding down the command key, press enter.

References.

Discrete Mathematics Through Applications, Crisler, Fisher, Froelich, Freeman
Contemporary Precalculus through applications, NCSSM, Everyday Learning