

Algebraic Manipulation

- **Boolean functions are made up of terms:**
 - The term consists of a number of literals.
 - The literal is a variable or the complement of a variable.
 - Each term is represented by a logic gate and each literal represents an input to a logic gate.
 - By reducing the number of terms, the number of literals, or both, a simpler logic circuit can be used to implement the Boolean function.
- **Reduction of the number of terms and/or number of literals is done by algebraic manipulation.**

Simplification Theorems

$$X Y + X Y' = X$$

$$(X + Y) (X + Y') = X$$

$$X + X Y = X$$

$$X (X + Y) = X$$

$$(X + Y') Y = X Y$$

$$X Y' + Y = X + Y$$

These are great for simplifying Boolean Expressions.

The trick is to find X and Y.

$$(X' + W + YZ) (W' + X' + YZ)$$

$$(W + \boxed{X' + YZ}) (W' + \boxed{X' + YZ})$$

$$X' + YZ$$

Multiplying Out

- Form an expression which is a sum of products
- All terms are products of single variables only
- (no parentheses)

$A B C' + D E + F G H$ Yes

$A B + C D + E$ Yes

$A B + C (D + E)$ No

Example:

$(A' + B)(A' + C)(C + D)$

Use $(X + Y)(X + Z) = X + Y Z$

$(A' + B C)(C + D)$

Multiply Out

$A' C + A' D + B C C + B C D$

Use $X \cdot X = X$

$A' C + A' D + B C + B C D$

Use $X + X Y = X$

$A' C + A' D + B C$

Factoring

- Form an expression which is a product of sums
- Final form is products only
- All sum terms are single variables only

$(A + B + C')(D + E)$ Yes

$(A + B)(C + D')E'F$ Yes

$(A + B + C')(D + E) + H$ No

$(A' + BC)(D + E)$ No

Example:

$AB + CD$

Use $X + YZ = (X + Y)(X + Z)$

$(AB + C)(AB + D)$

Use $X + YZ = (X + Y)(X + Z)$ again

$(A + C)(B + C)(AB + D)$

And again

$(A + C)(B + C)(A + D)(B + D)$

Factoring – Another Example

$$A B C + A B' C + C' D'$$

$$(A + A B' C + C' D')(B C + A B' C + C' D')$$

$$(A + A B' C + C' D')(B + A B' C + C' D')(C + A B' C + C' D')$$

NO! Think First

$$A B C + A B' C + C' D'$$

$$\text{Use } X Y + X Y' = X$$

$$A C + C' D'$$

$$\text{Use } X + Y Z = (X + Y)(X + Z)$$

$$(A C + C')(A C + D')$$

$$\text{Use } X Y' + Y = X + Y$$

$$(A + C')(A C + D')$$

$$\text{Use } X + Y Z = (X + Y)(X + Z) \text{ again}$$

$$(A + C')(A + D')(C + D')$$

Inversion

Inversion or Complement of a Function means all 0 outputs become 1 and all 1 outputs become 0

DeMorgan's Laws

$$(X + Y)' = X' Y'$$

$$(X Y)' = X' + Y'$$

Proof:

X	Y	X'	Y'	X + Y	(X + Y)'	X'Y'	X Y	(XY)'	X' + Y'
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

For N variables:

$$(X_1 + X_2 + X_3 + \dots + X_n)' = X_1' X_2' X_3' \dots X_n'$$

$$(X_1 X_2 X_3 \dots X_n)' = X_1' + X_2' + X_3' + \dots + X_n'$$

DeMorgan's Laws

For complex expressions,
apply DeMorgan's Laws successively

Example:

$$\begin{aligned}& [(a'b + 1)(cd + e' + 0)]' \\& (a'b + 1)' + (cd + e' + 0)' \\& [(a'b)' \cdot 1'] + (cd)' \cdot (e')' \cdot 0' \\& (a + b') \cdot 0 + (c' + d') \cdot e \cdot 1\end{aligned}$$

Note: Expression is not simplified

DeMorgan's Laws - One Step Rule

$$[f(X_1, X_2, \dots X_N, 0, 1, +, \bullet)]' = f(X_1', X_2', \dots X_N', 1, 0, \bullet, +)$$

Replace all variables with the inverse

Replace + with $\bullet$ and $\bullet$ with +

Replace 0 with 1 and 1 with 0

Be careful of hierarchy ()

Beware - Danger!!

Note that $(XY)'$ is not equal to $X'Y'$

Here is how you will make this mistake:

Factor:

$$A B (C' + DE) + A' B' (C' + DE)$$

Use $X Y + X Y' = X$

$$(C' + DE)$$

Unfortunately, this is WRONG, but

$$A B (C' + DE) + (A' + B') (C' + DE)$$

can be factored this way.

Duality

To form the Dual of an expression:

- Replace OR with AND and AND with OR
(+ with • and • with +)
- Replace 0 with 1 and 1 with 0
- Be careful of hierarchy ()

$$[f (X_1, X_2, \dots X_N, 0, 1, +, \bullet)]^D = f (X_1, X_2, \dots X_N, 1, 0, \bullet, +)$$

Duality - Example

Find the dual of this expression (do not simplify):

$$\{[A \cdot 0 + B(C + DE') \cdot 1](D' + E \cdot 0) + F'\}^D$$

Work from the outside in

$$\{\{[A \cdot 0 + B(C + DE') \cdot 1](D' + E \cdot 0)\}^D\} \cdot F'$$

$$\{[A \cdot 0 + B(C + DE') \cdot 1]^D + (D' + E \cdot 0)^D\} \cdot F'$$

$$\{(A \cdot 0)^D \cdot [B(C + DE') \cdot 1]^D + (D' + E \cdot 0)^D\} \cdot F'$$

$$\{(A \cdot 0)^D \cdot [B(C + DE') \cdot 1]^D + [D' \cdot (E + 1)]\} \cdot F'$$

$$\{(A + 1) \cdot [B + (C + DE')^D + 0] + [D' \cdot (E + 1)]\} \cdot F'$$

$$\{(A + 1) \cdot [B + C(D + E') + 0] + [D' \cdot (E + 1)]\} \cdot F'$$

Duality

- If two Boolean Expressions are equal, the duals are equal
- Another way to form the dual of an expression is to
 - Find the inverse
 - Replace all variables with their complements

Two More Boolean Algebra Theorems

$$(X + Y)(X' + Z) = XZ + X'Y$$

Take the Dual of this expression:

$$\begin{aligned} [(X + Y)(X' + Z)]^D &= [XZ + X'Y]^D \\ [(X + Y)^D + (X' + Z)^D] &= (XZ)^D (X'Y)^D \\ XY + X'Z &= (X + Z)(X' + Y) \end{aligned}$$

Note that the Dual of this expression is
(almost) the same expression.

This Theorem is very useful for factoring and multiplying out.